Modeling Damage Induced by Deviatoric Stress in Rock: Theoretical Framework

Hao Xu[1,] and Chloé Arson[2]

[1]Ph.D student, Damage Poro-Mechanics Lab, School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, Georgia, USA; PH (979) 777-9511; email: haoxu@gatech.edu
[2]Assistant Professor; Ph.D; Damage Poro-Mechanics Lab, School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, Georgia, 30332-0620; PH (404) 385-0143; email: chloe.arson@ce.gatech.edu

ABSTRACT

This work focuses on the development of a new anisotropic damage model for porous rocks. This damage model is formulated within the framework of thermodynamic irreversible processes. Flow rules are expressed with the energy release rate conjugate to damage, as opposed to stress in most rock damage models. The damage criterion is chosen so as to distinguish between tension and compression strength. Non-elastic deformation due to damage is computed by an associate flow rule to capture the development of crack-induced strains in the main directions of damage. Preliminary numerical results illustrate the potential for a crack to propagate under hydraulic pressure. This research work is expected to link damage mechanics with fracture propagation, for possible applications of hydraulic fracturing at reservoir scale.

INTRODUCTION

Continuum Damage Mechanics (CDM) initially aimed to predict deformation and stiffness in solids subject to cracking. The applications of CDM vary from metals to quasi-brittle geomaterials. Geomaterials have a heterogeneous porous structure needing rigorous characterization. Porous networks are generally complex, especially in microporous rock such as coal and shale, which comprise flaws ranging from the nano-scale to the millimeter scale. Damage effects are often analyzed at the “meso-scale”, i.e. at the scale of a Representative Elementary Volume Element (REV): this avoids to solve the difficulty in modeling cracks at the micro-scale (with non-uniform dietribution of micro-cracks), and its framework is well-suited for numerical implementation in Finite Element Methods (FEM).
In general, dissipative processes can be modeled by considering (1) the transformation of a single constituent (at the molecule or at the pore scale), (2) the response of an aggregate of individual discontinuities (pores and cracks of the same scale for instance), or (3) the behavior of a REV containing a multi-scale system of various species and phases. In CDM, discontinuities are modeled as energy losses at the scale of a REV

representing a continuum. CDM could be used to model the damaged zone ahead of the fracture tip, which would be of interest for rock subject to hydraulic fracturing or shear faulting for instance. It raises many issues related to the difficult modeling of the transition between damaged continuum (small discontinuities like cracks) and discontinuous medium (larger discontinuities hydraulic fractures or faults) (Mazars and Pijaudier-Cabot, 1996). Several methods were proposed, including an average damage computation at the scale of a REV (Valkó and Economides, 1994), and coupled numerical algorithms.

The goal of this research work is to develop a damage model accounting for crack-induced anisotropy induced by deviatoric stress in quasi-brittle geomaterials. The first section reviews the main strategies adopted so far to model anisotropy in geomaterials. The following part presents the thermodynamic framework and the main assumptions of the proposed anisotropic damage model. The last section presents a preliminary numerical study on stress redistributions around a hydraulic fracture, performed with a Finite Element code in the elastic domain of the proposed anisotropic damage model.

STATE OF THE ART: ANISOTROPIC DAMAGE MODELS

In geomechanics, the anisotropic damage variable is usually a second-order tensor which can be viewed as Kachanov's crack density tensor (Kachanov, 1958):

$$\boldsymbol{D} = \sum_{k=1}^{N} d_k \boldsymbol{n}_k \otimes \boldsymbol{n}_k$$

(in which the REV is assumed to contain N cracks characterized by a normal direction $\boldsymbol{n}_k$ and a volumetric fraction d_k), or as Oda's fabric tensor (Oda, 1984):

$$\boldsymbol{F} = \frac{1}{V_{REV}} \int_0^\infty \int_\Omega E\left(r, \boldsymbol{n}\right) d\boldsymbol{n} dr$$

(in which $E\left(r, \boldsymbol{n}\right)$ is the mathematical expectancy of a crack of radius r and normal direction $\boldsymbol{n}$ in the REV V_{REV}). For instance, Cicekli *et al.* (2007), Murakami and Kamiya (1996) used a second-order damage tensor in a free energy potential expressed in terms of elastic strains or modified strains. Chaboche (1993) and Pellet *et al.* (2005) introduces a parameter expressing the degree of anisotropy, allowing accounting for non-orthotropic damage. An anisotropic damage model based on a stress-dependent free energy potential was proposed by Shao *et al.* (2005), and Zhou *et al.* (2006), while Hayakawa and Murakami (1997) used a modified stress tensor to account for the difference between compressive and tensile stress. The damage evolution law derives from a criterion expressed in terms of the energy release rate conjugate to damage. The main limitations of these anisotropic damage models currently used in geomechanics are:

- *The difficult expression of a flow rule for anisotropic damage.* The damage criterion is not expressed in terms of the energy release rate ($\boldsymbol{Y}$) conjugate to damage, but rather with a positive projection part (noted $\boldsymbol{Y}^+$). As a result, damage evolution law is generally not a true associate flow rule. It exhibits a non-smooth damage surface (in general, several branches with sharp connections) and causes computation difficulty in numerical methods.

- *The difficult account for possible damage rotation.* The previous anisotropic CDM models generally assume that the principal directions of the damage tensor correspond to the principal directions of stress or strain. However, crack opening and closure induce shear effects that affect material stiffness and make it difficult to ensure thermodynamic consistency. In addition, shear rotates the principal bases of stress and strain.

A NEW MODEL OF DAMAGE INDUCED BY DEVIATORIC STRESS

Thermodynamic Framework

Crack propagation opens material surfaces, which dissipates energy in the rock. Associated damage is modeled herein within the framework of hyper-elasticity:

$$\boldsymbol{\sigma} = \frac{\partial \psi_s(\boldsymbol{D}, \boldsymbol{\epsilon}^E)}{\partial \boldsymbol{\epsilon}^E} = \mathbb{C}_{\mathbb{e}}(\boldsymbol{D}) : \boldsymbol{\epsilon}^E$$

In which ψ_s is the Helmholtz free energy of the solid skeleton, which is a function with quadratic terms in the total elastic strain $\boldsymbol{\epsilon}^E$ and linear term in damage tensor $\boldsymbol{D}$; $\mathbb{C}_{\mathbb{e}}$ is the damaged elastic stiffness tensor. The thermodynamic conjugation relationship with damage writes:

$$\boldsymbol{Y} = -\frac{\partial \psi_s}{\partial \boldsymbol{D}}$$

$\boldsymbol{Y}$ is the damage driving force conducted to damage tensor $\boldsymbol{D}$. The free energy potential can also be expressed in Gibbs free energy, $G_s(\boldsymbol{\sigma}, \boldsymbol{D})$, which is in terms of stress and is obtained through a Legendre transform from Helmholtz free energy.

$$\psi_s\left(\boldsymbol{\epsilon}^E, \boldsymbol{D}\right) + G_s\left(\boldsymbol{\sigma}, \boldsymbol{D}\right) = \boldsymbol{\sigma} : \boldsymbol{\epsilon}^E$$

It is proposed instead to account for irreversible remaining crack openings induced by damage (Abu Al-Rub and Voyiadjis, 2003). The total deformation tensor is split into:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{el} + \boldsymbol{\epsilon}^{ed} + \boldsymbol{\epsilon}^{id}$$

In which $\boldsymbol{\epsilon}^{el}$ is the purely elastic deformation, $\boldsymbol{\epsilon}^{ed}$ is the elastic damage-induced deformation due to the degradation of mechanical stiffness, and $\boldsymbol{\epsilon}^{id}$ is the irreversible deformation tensor (cf. Fig. 1). $\boldsymbol{\sigma}_R$ in Fig. 1 is the compression stress needed to come back to a state of zero deformation ("residual stress").

Three functionals are needed to close the formulation (Collins & Houlsby, 1997):

- *Solid skeleton Helmholtz free energy* ψ_s *or Gibbs free energy* G_s, to get the stress/strain conjugation relationships.
- *A damage criterion* f_d, to predict the occurrence of damage.
- *A dissipation potential* g_d, to derive the evolution laws of internal variables.

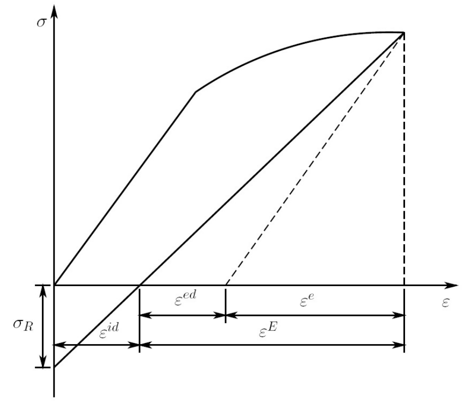


Figure 1. Decomposition of Strain.

Postulate 1: Expression of the Free Energy

In order to better account for states of tensile deformation under differential stress, it is proposed to postulate a free energy potential expressed in stress (Gibbs free energy, G_s). The expression of the free energy should have at most quadratic terms in $\boldsymbol{\sigma}$ (Halm and Dragon, 1998; Shao *et al.*, 2005). In addition, it is usually assumed that G_s is linear in $\boldsymbol{D}$. The expression proposed by (Shao *et al.*, 2005) is retained herein:

$$\begin{aligned} G_s\left(\boldsymbol{\sigma},\boldsymbol{D}\right) = \frac{1}{2}\boldsymbol{\sigma} : \mathbb{S}_0 : \boldsymbol{\sigma} + a_1 Tr\boldsymbol{D}\left(Tr\boldsymbol{\sigma}\right)^2 + a_2 Tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{D}) \\ + a_3 Tr\boldsymbol{\sigma}\, Tr(\boldsymbol{D}\cdot\boldsymbol{\sigma}) + a_4 Tr\boldsymbol{D}\, Tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) \end{aligned} \tag{1}$$

in which $\mathbb{S}_0$ is the compliance of undamaged material. The material parameters a_i need to be calibrated by experiments. The stress/strain relationship writes from Eq. 1:

$$\begin{aligned} \boldsymbol{\epsilon}^E = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{id}(\boldsymbol{D}) = \frac{\partial G_s}{\partial \boldsymbol{\sigma}} = \frac{1+\nu_0}{E_0}\boldsymbol{\sigma} - \frac{\nu_0}{E_0}(Tr\boldsymbol{\sigma})\,\boldsymbol{\delta} + 2a_1(Tr\boldsymbol{D}\,Tr\boldsymbol{\sigma})\,\boldsymbol{\sigma} \\ + a_2(\boldsymbol{\sigma}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{\sigma}) + a_3[\,Tr(\boldsymbol{\sigma}\cdot\boldsymbol{D})\,\boldsymbol{\delta} + (Tr\boldsymbol{\sigma})\,\boldsymbol{D}\,] + 2a_4(Tr\boldsymbol{D})\,\boldsymbol{\sigma} \end{aligned}$$

where $\boldsymbol{\delta}$ is the second-order identity tensor, and E_0 and ν_0 are undamaged Young's modulus and Poisson's ratio. The damage driving force conjugate to damage writes:

$$\boldsymbol{Y} = -\frac{\partial \psi_s}{\partial \boldsymbol{D}} = \frac{\partial G_s}{\partial \boldsymbol{D}} = a_1(Tr\boldsymbol{\sigma})^2\,\boldsymbol{\delta} + a_2\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + a_3 Tr(\boldsymbol{\sigma})\boldsymbol{\sigma} + a_4 Tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})\boldsymbol{\delta} \tag{2}$$

Postulate 2: Damage Function

The damage criterion is sought in a form similar to Drucker-Prager plasticity yield criterion, in order to predict the effects of stress difference. The first attempt, consisting in replacing stress by the damage driving force defined in Equation 2 shows that damage thresholds are the same in tension and in compression, as illustrated by the symmetries exhibited by the damage surface shown in Fig. . This is not realistic in materials like rock, which have more compression strength than tension strength. To overcome this limitation, it is proposed to use a more physician damage-driving force in the damage criterion. The modified expression of the damage function f_d is written in the following form:

$$f_d = \sqrt{J^*} - \alpha I^* - k \tag{3}$$

J^* and I^* are defined as:

$$J^* = \frac{1}{2}(\mathbb{P}_1 : \boldsymbol{Y} - \frac{1}{3}I^*\boldsymbol{\delta}) : (\mathbb{P}_1 : \boldsymbol{Y} - \frac{1}{3}I^*\boldsymbol{\delta}), \quad I^* = (\mathbb{P}_1 : \boldsymbol{Y}) : \boldsymbol{\delta}$$

$$\mathbb{P}_1(\boldsymbol{\sigma}) = \sum_{p=1}^{3} \left[H(\sigma^{(p)}) - H(-\sigma^{(p)})\right] \boldsymbol{n}^{(p)} \otimes \boldsymbol{n}^{(p)} \otimes \boldsymbol{n}^{(p)} \otimes \boldsymbol{n}^{(p)}$$

In which α is material parameter; $\mathbb{P}_1$ is a projection tensor. $H(\cdot)$ is the Heaviside distribution function, $\boldsymbol{\sigma}^{(p)}$ is the p^{th} eigenstress value, and $\boldsymbol{n}^{(p)}$ is the vector alined with the p^{th} principal direction of stress.

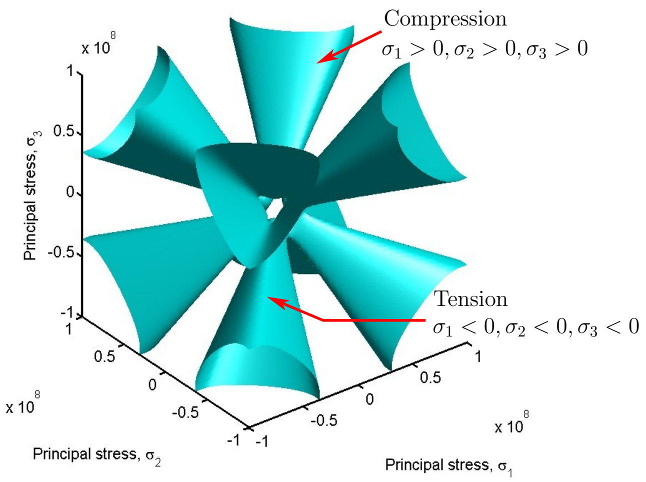
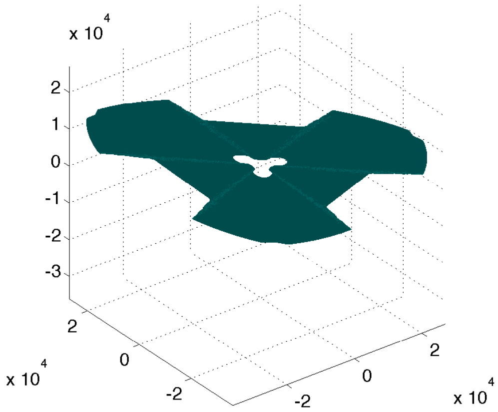


Figure 2. Free energy G_s obtained by replacing stress by the damage driving force in Drucker-Prager ’s yield function.

The threshold k in Eq. 3 is defined as a linear function of damage (Halm and Dragon, 1998):

$$k = C_0 - C_1 Tr\,(\boldsymbol{D})$$

The projection tensor $\mathbb{P}_1$ ensures that the eigenvalues of the “physical damage driving force tensor” ($\mathbb{P}_1 : \boldsymbol{Y}$) be of the same sign as the stress eigenvalues. In $\mathbb{P}_1 : \boldsymbol{Y}$ space, the damage surface is a cone. The plots of the damage surface in the space of the thermodynamic damage driving force $\boldsymbol{Y}$ (Fig. 3), and in stress space (Fig. 4), show that the damage surface is locally convex but globally non convex. Algorithms were proposed to solve the problem in numerical codes (Carstensen *et al.*, 2002; Pedroso *et al.*, 2008). Note that surface convexity is a sufficient but not necessary condition to satisfy the positivity of the dissipation potential: the thermodynamic framework is indeed consistent as long as the damage rate is non-negative (Desmorat, 2006).



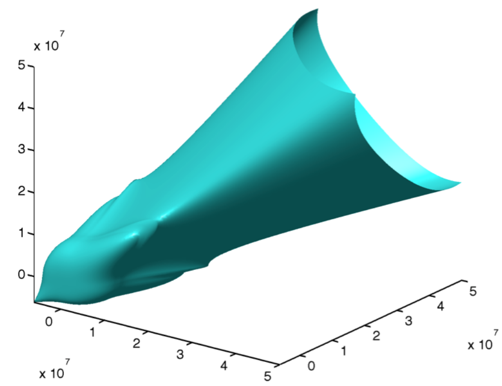


Figure 3. Damage surface in Y space. Figure 4. Damage surface in $\boldsymbol{\sigma}$ space.

Postulate 3: Expression of the Damage Potential

It is proposed to define the damage potential as a homogeneous function of degree one in $\boldsymbol{Y}$ (Collins and Houlsby, 1997):

$$g_d = \sqrt{\frac{1}{2}(\mathbb{P}_2 : \boldsymbol{Y}) : (\mathbb{P}_2 : \boldsymbol{Y})} - C_2 \tag{4}$$

The projection tensor $\mathbb{P}_2$ is introduced to represent the direction of damage rate (Fig. 5):

$$\mathbb{P}_2 = \sum_{p=1}^{3} H\left[\max_{q=1}^{3}(\sigma^{(q)}) - \sigma^{(p)}\right] \boldsymbol{n}^{(p)} \otimes \boldsymbol{n}^{(p)} \otimes \boldsymbol{n}^{(p)} \otimes \boldsymbol{n}^{(p)}$$

In the space of the "physical damage driving force" $\mathbb{P}_2 : \boldsymbol{Y}$, the surface of the damage potential is an octant of a sphere. In stress space, the shape of the damage surface exhibits discontinuities (Fig. 6 and 7). Damage obtained for basic loading paths show that with the dissipation potential defined in Eq. 4, it is possible to calibrate the material parameters a_i in order to ensure the positiveness of the components of $\frac{\partial g_d}{\partial \boldsymbol{Y}}$. Positivity of $\frac{\partial g_d}{\partial \boldsymbol{Y}}$ ensures the positivity of the damage rate, and therefore, the thermodynamic consistency of the model.

Postulate 4: Irreversible Deformation Flow Rule

Instead of deriving the rate of irreversible deformation from the potential (Eq. 4), the evolution law of irreversible strain is derived from an associate flow rule:

$$\dot{\boldsymbol{\epsilon}}^{id} = \dot{\lambda}_d \frac{\partial f_d}{\partial \boldsymbol{\sigma}} = \dot{\lambda}_d \frac{\partial f_d}{\partial \boldsymbol{Y}} \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{\sigma}}$$

CRACK HYDRAULIC PRESSURIZATION AND PROPAGATION

A parametric study on stress redistribution induced by crack pressurization was simulated with ABAQUS Finite Element program. The new model presented above was

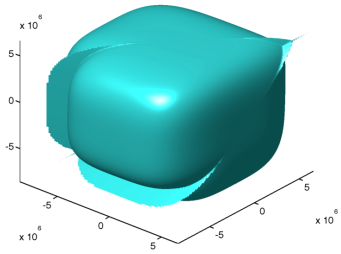


Figure 5. Damage potential in σ space.

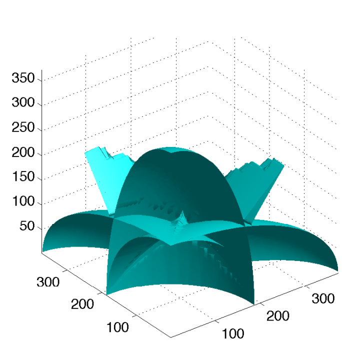


Figure 6. Damage potential in Y space.

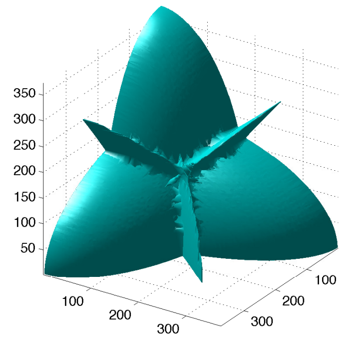


Figure 7. Damage potential in Y space.

programmed in a UMAT subroutine and the simulations were performed in the elastic domain (i.e. the damage model proposed was used for this simulation with ABAQUS UMAT, but material parameters were chosen so as to constrain the damage criterion to be negative). The purpose of this series of tests is to validate the algorithm in elasticity (against analytical solutions and numerical solutions obtained with purely elastic models built in ABAQUS). Further developments in the damage domain (when the damage function is non-negative) are expected to allow full-scale simulations of hydraulic fracturing in the future. Tests were performed for cylindrical samples, 100mm in height and 100mm in diameter. Three initial crack geometries were considered: (1) axis-symmetric simulation of half of a penny shaped crack (14mm in radius) perpendicular to the sample axis, (2) axis-symmetric simulation of half of a crack with a conic shape (14mm long and 2mm in diameter at the basis) oriented parallel to the sample axis, (3) non axis-symmetric simulation of the same ”conic crack” oriented perpendicular to the sample axis. In all cases, cracks were subjected to the same pressure (30MPa). Material parameters were those of a granite.

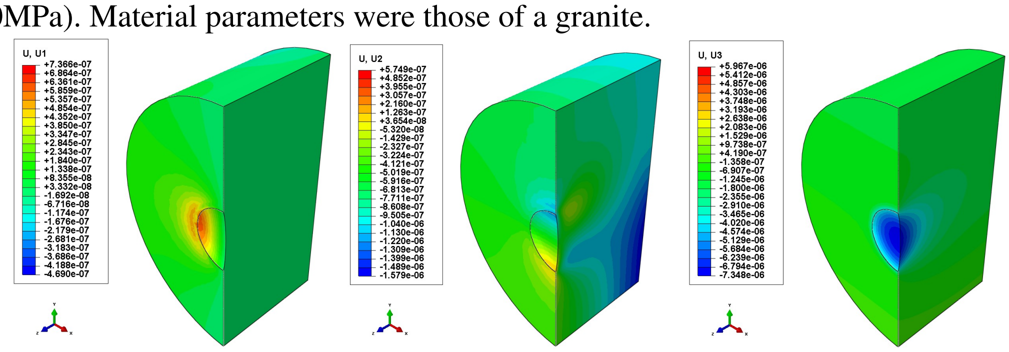


Figure 8. Axis-symmetric simulation of half of a penny shaped crack

For penny shape crack, the deformation around the crack tip is in the magnitude of 10^{-4}mm in both x- and y-direction. The difference between two is due to a lack of numerical accuracy. The aperture of the crack is one order magnitude lager (10^{-3}mm) than the radial expansion. For the axis-symmetric “conic crack”, the extension of the crack tip is in the order of 10^{-4}mm while the aperture of the crack is 10^{-4}mm in x-

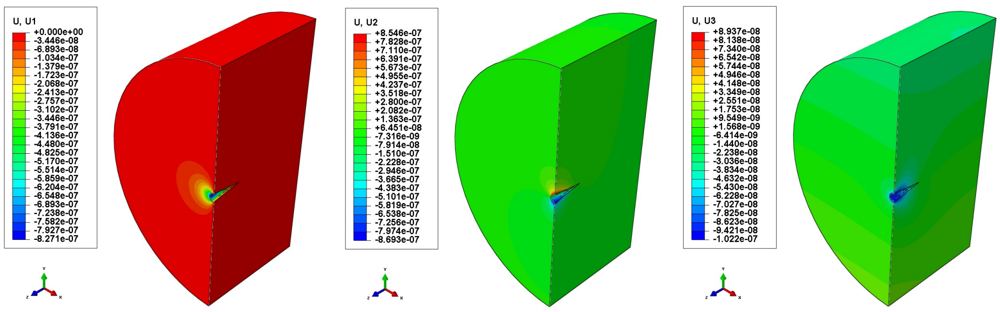


Figure 9. Axis-symmetric simulation of the “conic crack”

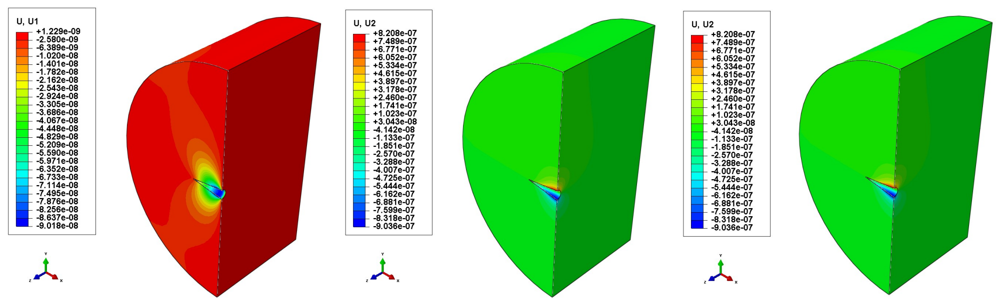


Figure 10. Non-Axis-symmetric simulation of the “conic crack”

direction, 10^{-5}mm in z-direction. Anisotropy of displacements is due to the position of the crack axis relative to the axis of the sample. The extension of the tip for the non-axis-symmetric “conic crack” in x-direction is in the magnitude of 10^{-5}mm, but the aperture is larger, 10^{-4}mm in y- and z-directions. Comparing cases 2 and 3: the same crack shape and pressure provide different results because different orientations produce different boundary effects in the sample. In case 1, the shape the characteristic crack length is the same but the penny shaped crack has a larger specific surface. As a result, pressurization provides more energy to the crack to propagate. The geometry of the sample and the shape of the initial crack influence crack propagation, which highlights the need for a fully anisotropic damage model to capture the effects of differential stress ahead of crack tips, and further capture the evolution of the damaged zone around hydraulic fractures.

CONCLUSION

A theoretical model is proposed within the frameworks of Continuum Damage Mechanics and thermodynamics of irreversible processes, in order to predict anisotropic damage induced by differential stress. The model distinguishes different damage thresholds in tension and compression. In order to relate damage evolution to differential stress, a damage criterion similar to Drucker-Prager yield function is defined - but it is expressed in terms of a damage driving force instead of stress. A non-associated flow rule is utilized for the damage evolution law, while an associated flow rule is employed for the irreversible strain due to residual crack opening. The damage increment is computed by deriving potentials from the total force conjugate to damage -

not from an absolute value of a part of force component. Within this set of assumptions, the new damage model meets thermodynamic requirements, follows a rigorous formulation and allows physically consistent predictions of damage, deformation and stiffness. Stress redistributions around a crack were simulated at the laboratory scale with a Finite Element code in the elastic domain of the proposed anisotropic damage model. Preliminary numerical results illustrate the potential for a crack to propagate under pressurization. Future work will be dedicated to optimization of fracture shape, modeling of multi-scale fracture propagation (fracture and smeared damage zone) and simulation of complex fracture networks from elementary sub-networks involving several basic crack shapes (such as penny shape, cone). Further developments are expected to provide new insights in the simulation of the damage zone ahead of the tip of hydraulic fractures.
This research work is the first step towards the development of a framework allowing modeling multi-scale crack propagation. Modeling outcomes are expected to link damage mechanics with fracture propagation, for possible applications in the understanding of faulting and hydraulic fracturing at reservoir scale.

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